Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Existence positive periodic solution of functional differential equation

By Xuanlong Fan
Qingdao Qiushi College,China

Abstract - The paper is concerned with functional differential equation

$$x'(t) = a(t)g(x(h_1(t)))x(t) - f\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x(t),$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T$, $g(x(h_1(t))) = diag\big(g_1(x_1(h_{11}(t))), \ldots, g_n(x_n(h_{1n}(t)))\big)$, $a(t) = diag(a_1(t), \ldots, a_n(t))$, $f\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big) = diag\Big(f_1(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v), \ldots, f_n(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v)\Big)^T$ *are periodic functions.*

Keywords : *Periodic solution; Functional differential equation; Fixed point; Cone.*

GJSFR-F Classication : FOR Code: 010109.

EXISTENCE POSITIVE PERIODIC SOLUTION OF FUNCTIONAL DIFFERENTIAL EQUATION

Strictly as per the compliance and regulations of :

Existence positive periodic solution of functional differential equation

Xuanlong Fan

Abstract - The paper is concerned with functional differential equation

$$x'(t) = a(t)g(x(h_1(t)))x(t) - f\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x(t),$$

where $x(t) = (x_1(t), \dots, x_n(t))^T$, $g(x(h_1(t))) = diag\big(g_1(x_1(h_{11}(t))), \dots, g_n(x_n(h_{1n}(t)))\big)$, $a(t) = diag(a_1(t), \dots, a_n(t))$, $f\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big) = diag\Big(f_1(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v), \dots, f_n(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v)\Big)^T$ are periodic functions.

Keywords : *Periodic solution; Functional differential equation; Fixed point; Cone.*

I. Introduction

The theory of differential systems have developed by mathematicians (see [1-5]). In this paper, we consider the following system

$$x'(t) = a(t)g(x(h_1(t)))x(t) - f\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x(t), \tag{1.1}$$

where

(H_1) $a_i\,(i = 1, \dots, n) \in C(\mathbb{R}, [0, +\infty))$ are T-periodic and there exists $t_1 \in (0, T)$ such that $a_i(t_1) > 0$;

(H_2) $h_{1i}\,(i = 1, \dots, n) \in C(\mathbb{R}, \mathbb{R})$ are p_1T-periodic, $h_{2i}\,(i = 1, \dots, n) \in C(\mathbb{R}, \mathbb{R})$ are p_2T-periodic and $h_{3i}\,(i = 1, \dots, n) \in C(\mathbb{R}, \mathbb{R})$ are p_3T-periodic;

(H_3) $g_i \in C([0, \infty), [0, \infty))$ are continuous, $0 < l_i \le g_i(u_i) < L_i < \infty$ for all $u_i > 0$, l_i, L_i are two positive constants. There exist positive constant $\mathbb{L}_i$ such that $\big|g_i(u_i) - g_i(v_i)\big| \le \mathbb{L}_i|u_i - v_i|$.

Author : *Department of Mathematics Qingdao Qiushi College Qingdao, Shandong 266111People's Republic of China.*
E-mail : *fanxuanlong@126.com*

Ref.

[1] A. Wan, D. Jiang, Existence of positive periodic solutions for functional differential equations, Kyushu J.Math. 56(1)(2002) 193-202.
[5] Wang. H, Positive periodic solutions of functional differential equations, J. Differential Equation 202(2004) 354-366.

(H_4) $f_i \in C(\mathbb{R} \times [0,\infty) \times [0,\infty) \times [0,\infty) \times \mathbb{R} \times \mathbb{R}, [0,\infty))$ are continuous functions. There exist positive functions $\alpha_{ij}(t) < +\infty$, $\beta_{ij}(t) < +\infty$, such that

$$\begin{aligned} & f_i\Big(t, u, \int_{-\varsigma}^{0} k(v)u(t-v)\mathrm{d}v, u', \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)u'(t-v)\mathrm{d}v\Big) \\ & -f_i\Big(t, v, \int_{-\varsigma}^{0} k(v)v(t-v)\mathrm{d}v, v', \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)v'(t-v)\mathrm{d}v\Big) \\ \leq\ & \sum_{j=1}^{n} \alpha_{ij}(t)|u_i - v_i| + \sum_{j=1}^{n} \beta_{ij}(t)|u_i' - v_i'|. \end{aligned}$$

Throughout this paper, a function is called ω-periodic($\omega > 0$)meaning ω is the least positive periodic of the function. Since p is the least positive rational number such that $\frac{p}{p_0}$, $\frac{p}{p_1}$, $\frac{p}{p_2}$ and $\frac{p}{p_3}$ are the positive integers, $pT = \omega$ is the least positive period of the periodic solutions of Eq.(1.1). System (1.1) contains many mathematical population models of delay differential equations [see(1-3,5,8-12)].

II. Preliminaries

In order to obtain the existence of a periodic solution of system (1.1), we then make the following preparations:

Let $\mathbb{E}$ be a Banach space and K be a cone in $\mathbb{E}$. The semi-order induced by the cone K is denoted by "$\leq$". That is, $x \leq y$ if and only if $y - x \in K$.

Let $\mathbb{E}$, $\mathbb{F}$ be two Banach spaces and $D \subset \mathbb{E}$, a continuous and bounded map $\Phi : \bar{\Omega} \to \mathbb{F}$ is called k-set contractive if for any bounded set $S \subset D$ we have

$$\alpha_{\mathbb{F}}(\Phi(S)) \leq k\alpha_{\mathbb{E}}(S).$$

Φ is called strict-set-contractive if it is k-set-contractive for some $0 \leq k < 1$.

The following lemma cited from Ref. [10,11] which is useful for the proof of our main results of this paper.

Lemma 2.1. [6,7] *Let K be a cone of the real Banach space X and $K_{r,R} = \{x \in K | r \leq ||x|| \leq R\}$ with $R > r > 0$. Suppose that $\Phi : K_{r,R} \to K$ is strict-set-contractive such that one of the following two conditions is satisfied:*

(i) $\Phi x \not\leq x, \quad \forall x \in K,\ ||x|| = r$ *and* $\Phi x \not\geq x, \quad \forall x \in K,\ ||x|| = R.$

(ii) $\Phi x \not\geq x, \quad \forall x \in K,\ ||x|| = r$ *and* $\Phi x \not\leq x, \quad \forall x \in K,\ ||x|| = R.$

Then Φ has at least one fixed point in $K_{r,R}$.

Remark 2.1. *Completely continuous operators are 0-set-contractive.*

In order to apply Lemma 2.1 to system (1.1), we consider the Banach space

$$C_\omega^0 = \{x(t) = (x_1(t), \ldots, x_n(t)) | x(t) \in C^0(\mathbb{R}, \mathbb{R}^n), x(t+\omega) = x(t),\ t \in \mathbb{R}\}$$

with the norm defined by $\|x\| = \sum_{i=1}^{n} |x_i|_0$, where $|x_i|_0 = \max_{t\in[0,\omega]}\{x_i(t)\}$, $i = 1, \ldots, n$ and

$$C_\omega^1 = \{x(t) = (x_1(t), \ldots, x_n(t)) | x(t) \in C^1(\mathbb{R}, \mathbb{R}^n), x(t+\omega) = x(t),\ t \in \mathbb{R}\}$$

Ref.
[3] D. Jiang, B. Zhang, Positive periodic solutions of functional differential equations and population models, Electron. J. Differential Equation 71(2002) 1-13.
[8] Y.K. Li, On a periodic neutral delay Lotka-Volterra system, Nonlinear Anal. 39 (2000) 767-778.

with the norm defined by $\|x\|_1 = \sum_{i=1}^{n} |x_i|_1,$, where $|x_i|_1 = \max\{|x_i|_0, |x_i'|_0\}, i = 1, \dots, n$. Then C_ω^0, C_ω^1 are all Banach space.

Let the map $\Phi = (\Phi_1, \dots, \Phi_n)$ be defined by

$$(\Phi x)(t) = \int_t^{t+\omega} G(t,s) f\Big(s, x(h_2(s)), \int_{-\varsigma}^{0} k(v)x(s-v)\mathrm{d}v, x'(h_3(s)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)x(s)\,\mathrm{d}s \tag{2.1}$$

for $x \in C_\omega^1, t \in \mathbb{R}$, where

$$G(t,s) = diag\big(G_1(t,s), \dots, G_n(t,s)\big),$$
$$G_i(t,s) = \frac{e^{-\int_t^s a_i(\theta)g_i(x_i(h_{1i}(\theta)))\,\mathrm{d}\theta}}{1 - e^{-\int_0^\omega a_i(\theta)g_i(x_i(h_{1i}(\theta)))\,\mathrm{d}\theta}},\ s \in [t, t+\omega],\ i = 1, \dots, n.$$

It is easy to see that $G(t+\omega, s+\omega) = G(t,s)$ and

$$\frac{\partial G(t,s)}{\partial t} = a(t)g(x(h_1(t)))G(t,s),$$
$$G(t+\omega, s+\omega) = G(t,s),$$
$$G(t, t+\omega) - G(t,t) = -I,$$
$$\frac{\sigma_i^{L_i}}{1-\sigma_i^{L_i}} \le G_i(t,s) \le \frac{1}{1-\sigma_i^{l_i}},\ \ s \in [t, t+\omega],$$

where $\sigma_i = e^{-\int_0^\omega a_i(\theta)\,\mathrm{d}\theta}$.

Define the cone K in X by

$$K = \Big\{x \Big| x \in C_\omega^1,\ x_i(t) \ge \delta_i |x_i|_1,\ t \in [0, \omega],\ i = 1, \dots, n\Big\},$$

where $0 < \delta < I$, $\delta = diag(\delta_1, \dots, \delta_n)$, $\delta_i = \dfrac{\sigma_i^{L_i}(1-\sigma_i^{l_i})}{1-\sigma_i^{L_i}}$.

Let

$$\xi_{1i} = \min\left\{\inf_{t\in R}\Big\{(\sigma_i^{l_i} - 1) + a_i(t)g_i(x_i(h_{1i}(t)))\Big\},\ \inf_{t\in R}\Big\{\frac{1-\sigma_i^{L_i}}{\sigma_i^{L_i}} - a_i(t)g_i(x_i(h_{1i}(t)))\Big\}\right\};$$

$$\xi_{2i} = \max\left\{\sup_{t\in R}\Big\{(\sigma_i^{l_i} - 1) + a_i(t)g_i(x_i(h_{1i}(t)))\Big\},\ \sup_{t\in R}\Big\{\frac{1-\sigma_i^{L_i}}{\sigma_i^{L_i}} - a_i(t)g_i(x_i(h_{1i}(t)))\Big\}\right\}.$$

(H_5) $0 < \xi_{1i} \le \xi_{2i} \le 1,\ i = 1, \dots, n.$

Lemma 2.2. *Assume that* $(H_1) - (H_5)$ *hold, then* Φ *maps* K *into* K.

Proof. For any $x \in K$, it is clear that $\Phi x \in C(R,R)$, we have

$$(\Phi x)(t+\omega) = \int_t^{t+\omega} G(t,s) f\Big(s, x(h_2(s)), \int_{-\varsigma}^{0} k(v)x(s-v)\mathrm{d}v, x'(h_3(s)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)x(s)\,\mathrm{d}s$$

$$\begin{aligned}
&= \int_t^{t+\omega} G(t+\omega, u+\omega) f\Big(u+\omega, x(h_2(u+\omega)), \int_{-\varsigma}^0 k(v)x(u+\omega-v)\mathrm{d}v, \\
&\quad x'(h_3(u+\omega)), \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)x'(u+\omega-v)\mathrm{d}v\Big) x(u+\omega)\,\mathrm{d}s \\
&= \int_t^{t+\omega} G(t,u) f\Big(u, x(h_2(u)), \int_{-\varsigma}^0 k(v)x(u-v)\mathrm{d}v, \\
&\quad x'(h_3(u)), \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)x'(u-v)\mathrm{d}v\Big) x(u)\,\mathrm{d}s \\
&= (\Phi x)(t).
\end{aligned}$$

Thus, $(\Phi x)(t+\omega) = (\Phi x)(t), t \in R$. So $\Phi x \in X$. For $x \in K, t \in [0,\omega]$, we have

$$\begin{aligned}
|\Phi_i x_i|_0 &\le \frac{1}{1-\sigma_i^{l_i}} \Big(\int_t^{t+\omega} f_i\Big(s, x(h_2(s)), \int_{-\varsigma}^0 k(v)x(s-v)\mathrm{d}v, x'(h_3(s)), \\
&\quad \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)x_i(s)\mathrm{d}s\Big), \; i=1,\dots,n
\end{aligned}$$

and

$$\begin{aligned}
(\Phi_i x_i)(t) &\ge \frac{\sigma_i^{L_i}}{1-\sigma_i^{L_i}} \Big(\int_t^{t+\omega} f_i\Big(s, x(h_2(s)), \int_{-\varsigma}^0 k(v)x(s-v)\mathrm{d}v, x'(h_3(s)), \\
&\quad \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)x_i(s)\mathrm{d}s\Big), \; i=1,\dots,n.
\end{aligned}$$

So we have $(\Phi_i x_i)(t) \ge \delta_i |\Phi_i x_i|_0$.

If $(\Phi_i x_i)'(t) \ge 0$, then

$$\begin{aligned}
(\Phi_i x_i)'(t) &= G_i(t, t+\omega) f_i\Big(t+\omega, x(h_2(t+\omega)), \int_{-\varsigma}^0 k(v)x(t+\omega-v)\mathrm{d}v, \\
&\quad x'(h_3(t+\omega)), \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)x'(t+\omega-v)\mathrm{d}v\Big)x_i(t+\omega) - G_i(t,t) f_i\Big(t, x(h_2(t)), \\
&\quad \int_{-\varsigma}^0 k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x_i(t) \\
&\quad + a_i(t) g_i(x_i(h_{1i}(t)))(\Phi_i x_i)(t) \\
&= -f_i\Big(t, x(h_2(t)), \int_{-\varsigma}^0 k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x_i(t) \\
&\quad + a_i(t) g_i(x_i(h_{1i}(t)))(\Phi_i x_i)(t) \\
&\le \Big((\sigma_i^{l_i}-1) + a_i(t) g_i(x_i(h_{1i}(t)))\Big)(\Phi x_i)(t) \le (\Phi_i x_i)(t), \; i=1,\dots,n. \qquad (2.2)
\end{aligned}$$

On the other hand, from (2.2), if $(\Phi_i x)'(t) < 0$, then

$$\begin{aligned}
-(\Phi_i x_i)'(t) &= f_i\Big(t, x(h_2(t)), \int_{-\varsigma}^0 k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x_i(t) \\
&\quad - a_i(t) g_i(x_i(h_{1i}(t)))(\Phi_i x_i)(t) \\
&\le \Big(\frac{1-\sigma_i^{L_i}}{\sigma_i^{L_i}} - a_i(t) g_i(x_i(h_{1i}(t)))\Big)(\Phi_i x_i)(t) \le (\Phi_i x_i)(t), \; i=1,\dots,n. \qquad (2.3)
\end{aligned}$$

Hence, $\Phi x \in K$. The proof of Lemma 2.2 is complete.

Notes

For convenience in the following discussion, we introduce the following notations:

$$\max_{t\in[0,\omega]}\{a_i(t)\} := a_i^M,$$

$$\max_{t\in[0,\omega]\, u\in B(0,R)} f_i\Big(s,u,\int_{-\varsigma}^{0}k(v)u(s-v)\mathrm{d}v,u',\int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)u'(s-v)\mathrm{d}v\Big) := \theta_i.$$

Notes

Lemma 2.3. *Assume that* $(H_1)-(H_5)$, *and* $\left(R\max_{t\in[0,\omega]}\left\{\sum_{i=1}^{n}\beta_i(t)\right\}\right) < 1$ *hold, then* $\Phi : K\bigcap\bar{\Omega}_R \to K$ *is strict-set-contractive, where* $\Omega_R = \{x\in C_\omega^1 : |x|_1 < R\}$.

Proof. It is easy to see that Φ is continuous and bounded. Now we prove that $\alpha_{C_\omega^1}(\Phi(S)) \le \left(R\max_{t\in[0,\omega]}\left\{\sum_{i=1}^{n}\beta_i(t)\right\}\right)\alpha_{C_\omega^1}(S)$ for any bounded set $S\subset\bar{\Omega}_R$. Let $\eta=\alpha_{C_\omega^1}(S)$. Then, for any positive number $\varepsilon < \left(R\max_{t\in[0,\omega]}\left\{\sum_{i=1}^{n}\beta_i(t)\right\}\right)\eta$, there is a finite family of subsets $\{S_i\}$ satisfying $S=\bigcup_i S_i$ with $\mathrm{diam}(S_i)\le\eta+\varepsilon$. Therefore

$$\|x_i-y\|_1 \le \eta+\varepsilon \quad \text{for any } x,y\in S_i. \tag{2.4}$$

As S and S_i are precompact in C_ω^0, it follows that there is a finite family of subsets $\{S_{ij}\}$ of S_i such that $S_i=\bigcup_j S_{ij}$ and

$$\|x-y\|\le\varepsilon \quad \text{for any } x,y\in S_{ij}. \tag{2.5}$$

Let $S\subset K$ be an arbitrary open bounded set in K, then there exists a number $R>0$ such that $\|x\|<R$ for any $x=(x_1,\ldots,x_n)^T\in S$. In fact, for any $x\in S$ and $t\in[0,\omega]$, we have

$$\begin{aligned}
|(\Phi_i x_i)(t)| &= \Big|\int_t^{t+\omega} G_i(t,s)x_i(s)f_i\Big(s,x(h_2(s)),\int_{-\varsigma}^{0}k(v)x(s-v)\mathrm{d}v,x'(h_3(s)),\\
&\qquad \int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)x'(s-v)\mathrm{d}v\Big)\mathrm{d}s\Big|\\
&\le \frac{1}{1-\sigma_i^{l_i}}\int_t^{t+\omega} x_i(s)f_i\Big(s,x(h_2(s)),\int_{-\varsigma}^{0}k(v)x(s-v)\mathrm{d}v,x'(h_3(s)),\\
&\qquad \int_{\widehat{\varsigma}}^{0}\widehat{k}(v)x'(s-v)\mathrm{d}v\Big)\mathrm{d}s\\
&\le \frac{R\omega}{1-\sigma_i^{l_i}}\max_{t\in[0,\omega]\, u_i\in B(0,R)} f_i\Big(s,u,\int_{-\varsigma}^{0}k(v)u(s-v)\mathrm{d}v,u',\\
&\qquad \int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)u'(s-v)\mathrm{d}v\Big)\Big) = \frac{\theta_i R\omega}{1-\sigma_i^{l_i}},\ i=1,\ldots,n
\end{aligned}$$

and

$$|(\Phi_i x_i)'(t)| = \Big|-f_i\Big(t,x(h_2(t)),\int_{-\varsigma}^{0}k(v)x(t-v)\mathrm{d}v,x'(h_3(t)),\int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x_i(t)$$

$$
\begin{aligned}
&+a_i(t)g_i(x_i(h_{1i}(t)))(\Phi_i x_i)(t)\Big| \\
\leq\ & \xi_{2i}\frac{\theta_i R\omega}{1-\sigma_i^{l_i}},\ i=1,\ldots,n.
\end{aligned}
$$

Hence

$$\|\Phi x\| \leq \sum_{i=1}^{n} H_i$$

and

$$\|(\Phi x)'\| \leq \sum_{i=1}^{n} \xi_{2i} H_i.$$

Applying the Arzela-Ascoli Theorem, we know that $\Phi(S)$ is precompact in C_ω^0. Then, there is a finite family of subsets $\{S_{ijk}\}$ of S_{ij} such that $S_{ij} = \bigcup_k S_{ijk}$ and

$$|(\Phi x)-(\Phi y)|_0 \leq \varepsilon \ \text{ for any } x,y\in S_{ijk}. \tag{2.6}$$

From (2.4), (2.5), (2.6), (H_3) and (H_4), for any $x,y\in S_{ijk}$, we obtain

$$
\begin{aligned}
&|(\Phi_i x_i)'-(\Phi_i y_i)'|_0 \\
=\ & \max_{t\in[0,\omega]}\Big\{\Big|a_i(t)g_i(x_i(h_{1i}(t)))(\Phi_i x_i)(t)-a_i(t)g_i(y_i(h_{1i}(t)))(\Phi_i y_i)(t) \\
&+f_i\Big(t,y(h_2(t)),\int_{-\varsigma}^{0}k(v)y(t-v)\mathrm{d}v,y'(h_3(t)),\int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)y'(t-v)\mathrm{d}v\Big)y(t) \\
&-f_i\Big(t,x(h_2(t)),\int_{-\varsigma}^{0}k(v)x(t-v)\mathrm{d}v,x'(h_3(t)),\int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x_i(t)\Big|\Big\} \\
\leq\ & \max_{t\in[0,\omega]}\{a_i(t)L|(\Phi_i x_i)(t)-(\Phi_i y_i)(t)|+a_i(t)\mathbb{L}_i|(\Phi_i x_i(t)|\big|x_i(t)-y_i(t)\big|\} \\
&+\max_{t\in[0,\omega]}\Big\{\Big|x_i(t)\Big[f_i\Big(t,y(h_2(t)),\int_{-\varsigma}^{0}k(v)y(t-v)\mathrm{d}v,y'(h_3(t)),\int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)y'(t-v)\mathrm{d}v\Big) \\
&-f_i\Big(t,x(h_2(t)),\int_{-\varsigma}^{0}k(v)x(t-v)\mathrm{d}v,x'(h_3(t)),\int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)x'(t-v)\mathrm{d}v\Big)\Big]\Big| \\
&+\big|x_i(t)-y_i(t)\big|f_i\Big(t,y(h_2(t)),\int_{-\varsigma}^{0}k(v)y(t-v)\mathrm{d}v,y'(h_3(t)),\int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)y'(t-v)\mathrm{d}v\Big)\Big\} \\
\leq\ & a_i^M L|(\Phi_i x_i)-(\Phi_i y_i)|_0+a_i^M\mathbb{L}_i\frac{\theta_i|x_i|_0\omega}{1-\sigma_i^{l_i}}\big|x_i(t)-y_i(t)\big|+\max_{t\in[0,\omega]}\Big\{\theta_i\big|x_i(t)-y_i(t)\big|\Big\} \\
&+|x_i|_0\Big(\sum_{i=1}^{n}\alpha_i(t)\big|x_i-y_i\big|_0+\sum_{i=1}^{n}\beta_i(t)\big|x_i'-y_i'\big|_1\Big) \\
&\leq \Big(a_i^M L+a_i^M\mathbb{L}_i\frac{\theta_i|x_i|_0\omega}{1-\sigma_i^{l_i}}+\theta_i+|x_i|_0\alpha_i(t)\Big)\varepsilon+|x_i|_0\beta_i(t)\Big(\eta+\varepsilon\Big)
\end{aligned}
$$

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

$$\leq \left(a_i^M L + a_i^M \mathbb{L}_i \frac{\theta_i |x_i|_0 \omega}{1-\sigma_i^{l_i}} + \theta_i + |x_i|_0 \alpha_i(t) + |x_i|_0 \beta_i(t)\right)\varepsilon + |x_i|_0 \beta_i(t)\eta,\ i = 1,\dots,n. \quad (2.7)$$

From (2.6) and (2.7), for any $x, y \in S_{ijk}$, we have

$$\begin{aligned} &\|\Phi x - \Phi y\|_1 \\ &\leq \left(a_i^M L + \max_{1\leq i\leq n}\left\{\sum_{i=1}^n a_i^M \mathbb{L}_i \frac{\theta_1 \omega}{1-\sigma_i^{l_i}}\right\} + \sum_{i=1}^n \theta_i + R\max_{t\in[0,\omega]}\left\{\sum_{i=1}^n \alpha_i(t)\right\}\right. \\ &\left. + R\max_{t\in[0,\omega]}\left\{\sum_{i=1}^n \beta_i(t)\right\}\right)\varepsilon + R\max_{t\in[0,\omega]}\left\{\sum_{i=1}^n \beta_i(t)\right\}\eta. \end{aligned}$$

As ε is arbitrary small, it follows that

$$\alpha_{C^1_\omega}(\Phi(S)) \leq \left(R\max_{t\in[0,\omega]}\left\{\sum_{i=1}^n \beta_i(t)\right\}\right)\alpha_{C^1_\omega}(S).$$

Therefore, Φ is strict-set-contractive. The proof of Lemma 2.3 is complete.

For convenience in the following discussion, we introduce the following notations:

$$\begin{cases} \lim\limits_{u\to 0} \sup \max_{t\in[0,\omega]} \dfrac{f_i\left(t,u,\int_{-\varsigma}^0 k(v)u(t-v)\mathrm{d}v, u', \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)u'(t-v)\mathrm{d}v\right)}{\sum_{i=1}^n u_i + \sum_{i=1}^n u_i'} = f_i^0, \\ \lim\limits_{u\to\infty} \inf \min_{t\in[0,\omega]} \dfrac{f_i\left(t,u,\int_{-\varsigma}^0 k(v)u(t-v)\mathrm{d}v, u', \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)u'(t-v)\mathrm{d}v\right)}{\sum_{i=1}^n u_i + \sum_{i=1}^n u_i'} = f_i^\infty. \end{cases} \quad (2.8)$$

III. Main Result

Our main result of this paper is as follows:

Theorem 3.1. *Assume that $(H_1)-(H_5)$ hold, then system (1.3) has at least one positive ω-periodic solution.*

Proof. According (2.8), for any

$$0 < \varepsilon < \min\left\{\frac{1}{2}, \frac{1}{4}\min_{1\leq i\leq n} f_i^\infty\right\},$$

there exist positive numbers $r_0 < R_0$ such that for $i = 1,\dots,n$,

$$\begin{aligned} &f_i\left(t,u,\int_{-\varsigma}^0 k(v)u(t-v)\mathrm{d}v, u', \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)u'(t-v)\mathrm{d}v\right) \\ &< (f_i^0+\varepsilon)\left(\sum_{i=1}^n u_i + \sum_{i=1}^n u_i'\right) \quad \text{for} \quad 0 < \sum_{i=1}^n |u_i|_1 < r_0 \end{aligned}$$

and

$$f_i\left(t,u,\int_{-\varsigma}^0 k(v)u(t-v)\mathrm{d}v, u', \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)u'(t-v)\mathrm{d}v\right)$$

$$> \left(f_i^\infty - \varepsilon\right)\left(\sum_{i=1}^n u_i + \sum_{i=1}^n u_i'\right) \quad \text{for} \quad \sum_{i=1}^n |u_i|_1 > R_0$$

Let

$$R = \max\left\{\left(\min_{1\le i\le n}\left\{\frac{2\sigma_i^{L_i}\omega\left(f_i^\infty - \varepsilon\right)}{1-\sigma_i^{L_i}}\delta_i^2\right\}\right)^{-1}, \min_{1\le i\le n}\left\{\delta_i^{-1}\right\}R_0\right\}$$

and

$$0 < r < \min_{1\le i\le n}\left\{\frac{1-\sigma_i^{l_i}}{2\omega\left(f_i^0+\varepsilon\right)}\delta_i,\, r_0\right\}.$$

Then we have $0 < r < R$. From Lemmas 2.2 and 2.3, we know that Φ is strict-set-contractive on $K_{r,R}$. In view of Lemma 2.1, we see that if there exists $x^* \in K$ such that $\Phi x^* = x^*$, then x^* is one positive ω-periodic solution of system (1.1).

First, we prove that $\Phi x \not\geq x$, $\forall x \in K$, $\|x\|_1 = r$. Otherwise, there exists $x \in K$, $\|x\|_1 = r$ such that $\Phi x \geq x$. So $\|x\| > 0$ and $\Phi x - x \in K$, which implies that

$$(\Phi_i x_i)(t) - x_i(t) \geq \delta_i|\Phi_i x_i - x_i|_1 \geq 0 \quad \text{for any } t \in [0,\omega]. \tag{3.1}$$

Moreover, for $t \in [0,\omega]$, we have

$$\begin{aligned}(\Phi_i x_i)(t) &= \int_t^{t+\omega} G_i(t,s)x_i(s)f_i\Big(s, x(h_2(s)), \int_{-\varsigma}^0 k(v)x(s-v)\mathrm{d}v,\\ &\qquad x'(h_3(s)), \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)\mathrm{d}s\\ &\leq \frac{1}{1-\sigma_i^{l_i}}|x_i|_0\left[2\omega\left(f_i^0+\varepsilon\right)\sum_{i=1}^n |x_i|_1\right]\\ &= \frac{2\omega\left(f_i^0+\varepsilon\right)}{1-\sigma_i^{l_i}}|x_i|_0 r\\ &< \delta_i|x_i|_0, \quad i = 1,\ldots,n. \end{aligned} \tag{3.2}$$

In view of (3.1) and (3.2), we have

$$\|x\| \leq \|\Phi x\| = \sum_{i=1}^n (\Phi_i x_i)|_0 < \max_{1\le i\le n}\{\delta_i\}\|x\| < \|x\|,$$

which is a contradiction. Finally, we prove that $\Phi x \not\leq x$, $\forall x \in K$, $\|x\|_1 = R$ also holds. For this case, we only need to prove that

$$\Phi x \not< x \quad x \in K,\ \|x\|_1 = R.$$

Suppose, for the sake of contradiction, that there exists $x \in K$ and $\|x\|_1 = R$ such that $\Phi x < x$. Thus $x - \Phi x \in K \setminus \{0\}$. Furthermore, for any $t \in [0,\omega]$, we have

$$x(t) - (\Phi x)(t) \geq \delta|x - \Phi x|_1 > 0. \tag{3.3}$$

In addition, for any $t \in [0,\omega]$, we find

$$(\Phi_i x_i)(t) = \int_t^{t+\omega} G_i(t,s)x_i(s)f_i\Big(s, x(h_2(s)), \int_{-\varsigma}^0 k(v)x(s-v)\mathrm{d}v,$$

Notes

Notes

$$
\begin{aligned}
&x'(h_3(s)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)\,\mathrm{d}s \\
&\geq \frac{\sigma_i^{L_i}}{1-\sigma_i^{L_i}}\delta_i|x_i|_1\left[2\omega\Big(f_i^{\infty}-\varepsilon\Big)\sum_{i=1}^{n}\delta_i|x_i|_1\right] \\
&\geq \frac{2\sigma_i^{L_i}\omega\Big(f_i^{\infty}-\varepsilon\Big)}{1-\sigma_i^{L_i}}\delta_i|x_i|_1 \min_{1\leq i\leq n}\{\delta_i\}\sum_{i=1}^{n}|x_i|_1, \quad i=1,\dots,n. \qquad (3.4)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\Phi x\|_0 &= \sum_{i=1}^{n}|(\Phi_i x_i)|_0 \geq \frac{2\sigma_i^{L_i}\omega\Big(f_i^{\infty}-\varepsilon\Big)}{1-\sigma_i^{L_i}}\delta_i|x_i|_1\sum_{i=1}^{n}|x_i|_1 \\
&\geq \min_{1\leq i\leq n}\left\{\frac{2\sigma_i^{L_i}\omega\Big(f_i^{\infty}-\varepsilon\Big)}{1-\sigma_i^{L_i}}\delta_i^2\right\}\sum_{i=1}^{n}|x_i|_1\sum_{i=1}^{n}|x_i|_1 \\
&\geq \min_{1\leq i\leq n}\left\{\frac{2\sigma_i^{L_i}\omega\Big(f_i^{\infty}-\varepsilon\Big)}{1-\sigma_i^{L_i}}\delta_i^2\right\}R^2 = R. \qquad (3.5)
\end{aligned}
$$

From (3.3) – (3.5), we obtain

$$\|x\| > \|\Phi x\| \geq R,$$

which is a contradiction. Therefore, conditions (i) and (ii) hold. By Lemma 2.2, we see that Φ has at least one nonzero fixed point in K. Therefore, system (1.3) has at least one positive ω-periodic solution. The proof of Theorem 3.1 is complete.

IV. Examples

Consider the following system [8]

$$x_i'(t) = x_i(t)\left[a_i(t) - \sum_{j=1}^{n}\alpha_{ij}(t)x_j(t-\tau_{ij}) - \sum_{j=1}^{n}\beta_{ij}(t)x_j'(t-\sigma_{ij})\right], \qquad (4.1)$$

where a_i, α_{ij}, β_{ij} $(i=1,\dots,n,\ j=1,\dots,n) \in \big(\mathbb{R},(0,+\infty)\big)$ are functions with periodic ω, τ_{ij}, σ_{ij} $(i=1,\dots,n,\ j=1,\dots,n) \in [0,+\infty)$ are constants.

Corollary 4.1. *Assumed* $(H_1)-(H_5)$ *and* $\max_{1\leq i\leq n}\left\{R\sum_{j=1}^{n}\beta_{ij}(t)\right\} < 1$ *hold, Eq.(4.1) has at least one ω-periodic solution.*

Proof. In this case

$$
\begin{aligned}
&f_i\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big) \\
= &\sum_{j=1}^{n}\alpha_{ij}(t)x_j(t-\tau_{ij}) + \sum_{j=1}^{n}\beta_{ij}(t)x_j'(t-\sigma_{ij}), \\
&g_i(x_i(h_1(t))) = 1, \\
&f_i^0 \leq \max_{1\leq i\leq n}\left\{\max_{t\in[0,\omega]}\Big\{\alpha_{ij}(t)\Big\} + \max_{t\in[0,\omega]}\Big\{\beta_{ij}(t)\Big\}\right\} < \infty, \ i=1,\dots,n
\end{aligned}
$$

and

$$f_i^{\infty} \leq \min_{1\leq i\leq n}\left\{\min_{t\in[0,\omega]}\left\{\alpha_{ij}(t)\right\}+\min_{t\in[0,\omega]}\left\{\beta_{ij}(t)\right\}\right\}>0,\ i=1,\dots,n.$$

It follows from Theorem 3.1 that system (4.1) has at least one positive periodic solution. The proof of Theorem 4.1 is complete.

Consider the following system [9]

$$x_i'(t)=x_i(t)\left[a_i(t)-\sum_{j=1}^{n}b_{ij}(t)\int_{-T_{ij}}^{0}K_{ij}(\theta)x_j(t+\theta)\mathrm{d}\theta-\sum_{j=1}^{n}c_{ij}(t)\int_{-\widehat{T}_{ij}}^{0}\widehat{K}_{ij}(\theta)x_j'(t+\theta)\mathrm{d}\theta\right],\quad(4.2)$$

where $a_i,\ b_{ij},\ c_{ij}\ (i=1,\dots,n,\ j=1,\dots,n)\in\big(\mathbb{R},(0,+\infty)\big)$ are functions with periodic ω, $T_{ij},\ \widehat{T}_{ij}\ (i=1,\dots,n,\ j=1,\dots,n)\in[0,+\infty)$, $K_{ij},\ \widehat{K}_{ij}\in(\mathbb{R},\mathbb{R}^+)$ satisfying $\int_{-T_{ij}}^{0}K_{ij}(\theta)x_j(t+\theta)\mathrm{d}\theta=1,\ \int_{-\widehat{T}_{ij}}^{0}\widehat{K}_{ij}(\theta)\mathrm{d}\theta=1,\ i,j=1,\dots,n.$

Corollary 4.2. *Assumed* $(H_1)-(H_5)$ *and* $\max_{1\leq i\leq n}\left\{R\sum_{j=1}^{n}c_{ij}(t)\right\}<1$ *hold, Eq.(4.1) has at least one* ω*-periodic solution.*

Proof. In this case

$$\begin{aligned}
&f_i\Big(t,x(h_2(t)),\int_{-\varsigma}^{0}k(v)x(t-v)\mathrm{d}v,x'(h_3(t)),\int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)x'(t-v)\mathrm{d}v\Big)\\
=&\sum_{j=1}^{n}b_{ij}(t)\int_{-T_{ij}}^{0}K_{ij}(\theta)x_j(t+\theta)\mathrm{d}\theta+\sum_{j=1}^{n}c_{ij}(t)\int_{-\widehat{T}_{ij}}^{0}\widehat{K}_{ij}(\theta)x_j'(t+\theta)\mathrm{d}\theta,\\
&g_i(x_i(h_1(t)))=1,\\
&f_i^{0}\leq\max_{1\leq i\leq n}\left\{\max_{t\in[0,\omega]}\left\{b_{ij}(t)\right\}+\max_{t\in[0,\omega]}\left\{c_{ij}(t)\right\}\right\}<\infty,\ i=1,\dots,n
\end{aligned}$$

and

$$f_i^{\infty}\leq\min_{1\leq i\leq n}\left\{\min_{t\in[0,\omega]}\left\{b_{ij}(t)\right\}+\min_{t\in[0,\omega]}\left\{c_{ij}(t)\right\}\right\}>0,\ i=1,\dots,n.$$

It follows from Theorem 3.1 that system (4.1) has at least one positive periodic solution. The proof of Theorem 4.1 is complete.

References Références Referencias

[1] A. Wan, D. Jiang, Existence of positive periodic solutions for functional differential equations, Kyushu J.Math. 56(1)(2002) 193-202.

[2] A. Wan, D. Jiang, X. Xu, A new existence theory for positive periodic solutions to functional differential equations, Comput. Math. Appl. 47(2004) 1257-1262.

[3] D. Jiang, B. Zhang, Positive periodic solutions of functional differential equations and population models, Electron. J. Differential Equation 71(2002) 1-13.

[4] Y. Li, Existence and global attractivity of a positive periodic solution fo a class of delay differential equation, Sci. China Ser. A 41(1998) 273-284.

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

[5] Wang. H, Positive periodic solutions of functional differential equations, J. Differential Equation 202(2004) 354-366.

[6] N.P. Các, J.A Gatica, Fixed point theorems for mappings in ordered Banach space, J. Math. Anal. Appl. 71(1979) 547-557.

[7] D. Guo, Positive solutions of nonlinear operator equations and its applications to nonlinear integral equations, Adv. Mtath. 13 (1984) 294-310(in chinese).

[8] Y.K. Li, On a periodic neutral delay Lotka-Volterra system, Nonlinear Anal. 39 (2000) 767-778.

[9] Y. Li, Positive periodic solutions of periodic neutral Lotka-Volterra system with distributed delays, Chaos, Solitons and Fractals. 2006, IN PRESS.

[10] Z. Yang, J. Cao, Positive periodic solutions of neutral Lotka-Volterra system with periodic delays, Comput. Math. Appl. 149 (2004) 661-687.

[11] G. Liu, J. Yan, Positive periodic solutions for a neutral delay Lotka-Volterra system. Nonlinear Anal. 67 (2007) 2642-2653.

[12] Z. Liu, L. Chen, Periodic solution of neutral LotkaCVolterra system with periodic delays, J. Math. Anal.Appl. 324(2006) 435-451.

[13] Y. Li, Positive periodic solutions of periodic neutral Lotka-Volterra system with state dependent delays, J. Math. Anal.Appl. 330 (2007) 1347-1362.

[14] Y. Li, Positive periodic solutions of discrete Lotka-Volterra competition systems with state dependent and distributed delays, Mathe. Compu,190(2007) 526-531.

Notes

Notes

This page is intentionally left blank